

Global well-posedness to the 3-D incompressible inhomogeneous Navier-Stokes equations with a class of large velocity

Cuili Zhai, Ting Zhang

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

In this article, we consider the global well-posedness to the 3-D incompressible inhomogeneous Navier-Stokes equations with a class of large velocity. More precisely, assuming $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^3) \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ for $p, q \in (1, 6)$ with $\sup(\frac{1}{p}, \frac{1}{q}) \leq \frac{1}{3} + \inf(\frac{1}{p}, \frac{1}{q})$, we prove that if $C\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}^{\alpha} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{-1+\frac{3}{p}}/\mu + 1) \leq 1$, $\frac{C}{\mu}(\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{-1+\frac{3}{p}} + \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1-\alpha} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{\alpha}) \leq 1$, then the system has a unique global solution $a \in \tilde{\mathcal{C}}([0, \infty); \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$, $u \in \tilde{\mathcal{C}}([0, \infty); \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$. It improves the recent result of M. Paicu, P. Zhang (J. Funct. Anal. 262 (2012) 3556-3584), where the exponent form of the initial smallness condition is replaced by a polynomial form.

Keywords: Inhomogeneous Navier-Stokes equations; Well-posedness; Littlewood-Paley theory.

2010 AMS Subject Classification: 35Q35, 76W05.

1 Introduction.

In this paper, we consider the global well-posedness of the following 3-D incompressible inhomogeneous Navier-Stokes equations with initial data in the critical Besov spaces

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu \mathcal{M}) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0, \end{cases} \quad (1.1)$$

where $\rho, u = (u_1, u_2, u_3)$ stand for the density and velocity field of the fluid respectively, $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function, and in general, the viscosity coefficient $\mu(\rho)$ is a smooth, positive function on $[0, \infty)$. Such system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [22] for the details derivation of this system.

When $\mu(\rho)$ is independent of ρ , i.e. μ is a positive constant, and ρ_0 is bounded away from 0, many authors showed their investigations on this system, see [6, 7, 20, 28] etc. Kazhikov [20] proved that (1.1) has a unique local smooth solution with regular initial data. In addition, they proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. However, the uniqueness of both type weak solutions has not been solved. Ladyzhenskaya and Solonnikov [21] first addressed the question of unique resolvability of (1.1). And recently, similar results were obtained by Danchin [12, 13] in \mathbb{R}^N with initial data in the almost critical spaces, and it [12] generalized the result by Fujita and Kato [16] denoted to the classical Navier-Stokes system.

In general, $\mu = \mu(\rho)$, under some special assumption, a lot of results about stability and well-posedness of Navier-Stokes equations were received by many authors, such as [1, 2, 3, 4, 5, 15, 22] etc. Diperna and Lions [15, 22] proved the global existence of weak solutions to (1.1) in any space dimensions. Yet the uniqueness and regularities of such weak solutions are big open questions even in two space dimension, as was mentioned by Lions in [22]. On the other hand, Abidi, Gui and Zhang [2] investigated the large time decay and stability to any given global smooth solutions of (1.1), which in particular implies the global well-posedness of 3-D inhomogeneous Navier-Stokes equations with axi-symmetric initial data and

without swirl for the initial velocity field provided that the initial density is close enough to a positive constant.

When the density ρ is away from zero, we denote by $a \stackrel{def}{=} \frac{1}{\rho} - 1$ and $\tilde{\mu}(a) \stackrel{def}{=} \mu(\frac{1}{1+a})$, then the system (1.1) can be equivalently reformulated as

$$(INS) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \operatorname{div}(2\tilde{\mu}(a)\mathcal{M})) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases}$$

In [1], Abidi proved if $1 < p < 2N$, $0 < \underline{\mu} < \tilde{\mu}(a)$, $u_0 \in \dot{B}_{p,1}^{\frac{N}{p}-1}(\mathbb{R}^N)$ and $a_0 \in \dot{B}_{p,1}^{\frac{N}{p}}(\mathbb{R}^N)$, then (INS) has a global solution provided that $\|a_0\|_{\dot{B}_{p,1}^{\frac{N}{p}}} + \|u_0\|_{\dot{B}_{p,1}^{\frac{N}{p}-1}} \leq c_0$ for some c_0 sufficiently small. Furthermore, the solution thus obtained is unique if $1 < p \leq N$. And this result generalized the corresponding results in [12, 13].

For simplicity, in this paper, we just take $\mu(\rho) = \mu$ and the space dimension $N = 3$. Thus (INS) becomes

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases} \quad (1.2)$$

Before we present our main result in this paper, let us recall the following results from Abidi, Paicu [5], Danchin, Mucha [14] and Paicu, Zhang [27]. We denote

$$E_{p,q,T} \stackrel{def}{=} \left\{ (a, u, \nabla \Pi) \left| \begin{array}{l} a \in \tilde{\mathcal{C}}_T(\dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)), u \in \tilde{\mathcal{C}}_T(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L_T^1(\dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)) \\ \nabla \Pi \in L_T^1(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \end{array} \right. \right\},$$

whereas

$$\tilde{\mathcal{C}}_T(\dot{B}_{p,r}^s(\mathbb{R}^3)) \stackrel{def}{=} \mathcal{C}([0, T]; \dot{B}_{p,r}^s(\mathbb{R}^3)) \cap \tilde{L}^\infty(0, T; \dot{B}_{p,r}^s(\mathbb{R}^3)).$$

For simplify, we denote $E_{p,q}$ when $T = \infty$.

Theorem 1.1 (see [5, 14]). *Let q, p satisfy $q, p \in (1, \infty)$ so that $\sup(\frac{1}{p}, \frac{1}{q}) \leq \frac{1}{3} + \inf(\frac{1}{p}, \frac{1}{q})$ and $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$. Let $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$, $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \leq c$ for some sufficiently small c , then the system (1.2) has a unique local solution $(a, u, \nabla \Pi)$ on $[0, T]$ such that $(a, u, \nabla \Pi) \in E_{p,q,T}$. Moreover, if $\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} \leq c'\mu$ for c' small enough, then the solution exists on $[0, +\infty)$.*

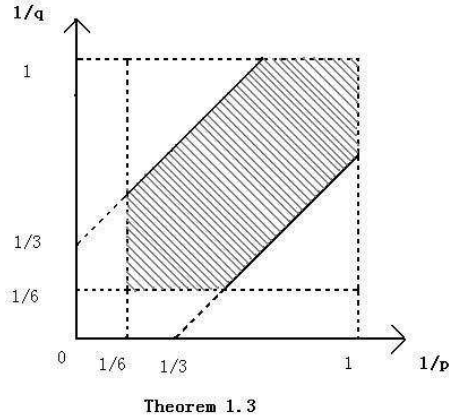
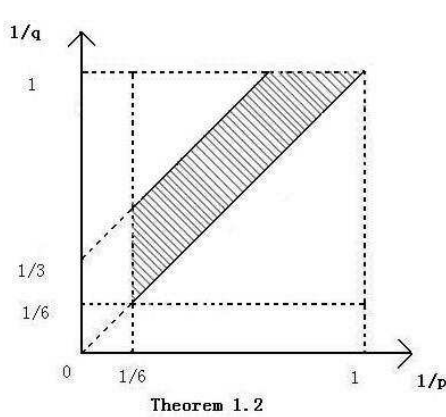
Indeed, Abidi and Paicu [5] only proved that the solution is unique when $\frac{1}{p} + \frac{1}{q} \geq \frac{2}{3}$, and very recently, Danchin and Mucha [14] improved the uniqueness for $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$ through lagrangian approach.

Motivated by [18, 26, 29], M. Paicu, P. Zhang proved the following theorem in [27] by applying the technology of a weighted Chemin-Lerner type norm.

Theorem 1.2 (see [27]). *Let $1 < q \leq p < 6$ with $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$. There exist positive constants c_0 and C_0 such that, for any data $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^3) \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ verifying*

$$\eta \stackrel{def}{=} (\mu \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}) \exp\{C_0 \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^2 / \mu^2\} \leq c_0 \mu, \quad (1.3)$$

the system (1.2) has a unique global solution $(a, u, \nabla \Pi) \in E_{p,q}$.



We note that, for the classical Navier-Stokes equations, the equation on u^3 is a linear system for fixed $u^h = (u_1, u_2)$, while the system on u^h is nonlinear. By using the technology of a weighted Chemin-Lerner type Besov spaces norm in [26] or Gronwall's inequality in [18, 29], the form of small initial conditions liking (1.3) were obtained, which implies that the third component of the initial velocity field can be large. And by using the algebraical structure of (1.2): $\operatorname{div} u = 0$, they [27] proved the Theorem 1.2 by energy estimates on the horizontal components and the vertical component of the velocity field respectively.

In this paper, we are going to relax the smallness condition in Theorems 1.1 and 1.2, so that (1.2) still has a unique global solution. Now we present the first main result in this paper:

Theorem 1.3. *Let p, q satisfy $p, q \in (1, 6)$ so that $\sup(\frac{1}{p}, \frac{1}{q}) \leq \frac{1}{3} + \inf(\frac{1}{p}, \frac{1}{q})$. There exists a positive constant C such that, for any data $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^3) \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ verifying*

$$C \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}^\alpha (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} / \mu + 1) \leq 1 \quad (1.4)$$

$$\frac{C}{\mu} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1-\alpha} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^\alpha) \leq 1, \quad (1.5)$$

whereas

$$\alpha = \begin{cases} \frac{1}{p}, & 1 < p < 5 \\ \varepsilon, & 5 \leq p < 6 \end{cases} \quad (1.6)$$

for $0 < \varepsilon < \frac{6}{p} - 1$, the system (1.2) has a unique global solution $(a, u, \nabla \Pi) \in E_{p,q}$.

Remark 1.1. Motivated by [30], using Gagliardo-Nirenberg inequality, we obtain $\|u^3\|_{L^\infty} \leq C \|u^3\|_{L_v^p}^{1-\frac{1}{p}} \|\operatorname{div}_h u^h\|_{L_v^p}^{\frac{1}{p}}$, and one can search for details in Lemma 2.4 in the second section. This implies that the velocity is large in one direction with the L^p framework functional space, but is small in the L^∞ framework functional space.

Remark 1.2. We assert that our theorem remains to be true in the case when the viscosity coefficient depends on the density by a regular function $\mu(\rho)$ with $\mu(\rho) \geq \mu > 0$. In this case, we just need a small modification of the proof to Theorem 1.3 by using the fact that: for any positive s , we have $\|\tilde{\mu}(a) - \tilde{\mu}(0)\|_{\dot{B}_{q,1}^s} \leq C(1 + \|a\|_{L^\infty})^{[s]+1} \|a\|_{\dot{B}_{q,1}^s}$, where $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\frac{1}{1+a})$.

Remark 1.3. We can also have a version of Theorem 1.3 in any space dimension. Just for a clear presentation, we choose to work in the three space dimension case here.

Remark 1.4. About ill-posedness of the classical incompressible Navier-Stokes equations with $\rho = 1$, Bourgain-Pavlovic [9] and Germain [17] proved the ill-posedness in the largest critical space $\dot{B}_{\infty,\infty}^{-1}$. Motivated by [9], Chen, Miao, Zhang [10] proved that the 3-D barotropic Navier-Stokes equations is ill-posed for the initial density and velocity belonging to the critical Besov spaces $(\dot{B}_{p,1}^{\frac{3}{p}} + \bar{\rho}, \dot{B}_{p,1}^{\frac{3}{p}-1})$ for $p > 6$, here $\bar{\rho}$ is a positive constant. In the future, we will work in ill-posedness of the incompressible inhomogeneous Navier-Stokes system in critical Besov spaces.

Very recently, Danchin and Mucha [14] also obtained a more general result by considering very rough densities in some multiplier spaces on the Besov spaces $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. In particular, they are able to consider the physical case of mixture of fluids with piecewise constant density. We emphasize that the main feature of the density used in this theorem is to be a multiplier on the velocity space. This allows to define the nonlinear terms containing products between the density and the velocity in the system (1.2). And Motivated by [14, 19], we can also replace the $\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}$ in the smallness condition (1.4) by $\|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}})}$ and prove a similar version of Theorem 1.3:

Theorem 1.4. *Let $\frac{3}{2} < p < 6$. Let $a_0 \in \mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))$ and $u_0 = (u_0^h, u_0^3) \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. Then there exists a positive constant C such that if*

$$\begin{aligned} C\|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}})}^\alpha (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} / \mu + 1) &\leq 1 \\ \frac{C}{\mu} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^{1-\alpha} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^\alpha) &\leq 1, \end{aligned} \quad (1.7)$$

whereas α defined as (1.6), then (1.2) has a unique global solution $a \in \tilde{\mathcal{C}}^\infty(\mathbb{R}^+; \mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)))$ and $u \in \tilde{\mathcal{C}}([0, \infty); \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$.

Remark 1.5. Using basic continuity results for the paraproduct operator (see [8]), one can obtain that any space $L^\infty(\mathbb{R}^3) \cap \dot{B}_{q,\infty}^{\frac{3}{q}}(\mathbb{R}^3)$ with q satisfying

$$\frac{1}{q} + \frac{1}{p} > \frac{1}{3} \quad \text{and} \quad \frac{1}{p} - \frac{1}{q} < \frac{1}{3} \quad (1.8)$$

embeds $\mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))$. It contains characteristic functions of C^1 -bounded domains whenever $p > 2$ (see the proof of Lemma A.7 in [14]). Hence our result applies to a mixture of fluids, which is of great physical interest.

Scheme of the proof and organization of the paper. In the second section, we shall collect some basic facts on Littlewood-Paley analysis. In the third section, we prove theorem 1.3 in case when $p \leq q$. Finally in the last section, we shall prove the case of $p > q$. And in the Appendix, we give the proof of Theorem 1.4.

Let us complete this section with the notations we are going to use in this context.

Notations. Let A, B be two operators, we denote $[A; B] = AB - BA$, the commutator between A and B . For $a \lesssim b$, we mean that there is a uniform constant C , which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b . $(d_j)_{j \in \mathbb{Z}}$ will be a generic element of $\ell^1(\mathbb{Z})$ so that $d_j \geq 0$ and $\sum_{j \in \mathbb{Z}} d_j = 1$.

For X a Banach space and I an interval of \mathbb{R} , we denote by $\mathcal{C}(I, X)$ the set of continuous functions on I with values in X , and by $L^p(I; X)$ stands for the set of measurable functions on I with values in X , such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$. Finally we denote $L_T^p(L_h^q(L_v^r))$ the space $L^p([0, T]; L^q(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^r(\mathbb{R}_{x_3})))$.

2 Preliminaries

The proof of Theorem 1.3 requires the Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^3$ (see e.g. [8]). Let φ be a smooth function supported in the ring $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for } |\xi| \neq 0.$$

Then for $u \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$\forall j \in \mathbb{Z}, \quad \Delta_j u \stackrel{\text{def}}{=} \varphi(2^{-j}D)u \quad \text{and} \quad S_j u \stackrel{\text{def}}{=} \sum_{\ell \leq j-1} \Delta_\ell u.$$

The Besov space can be characterized in virtue of the Littlewood-Paley decomposition. Let $\mathcal{S}'_h(\mathbb{R}^3)$ be the space of tempered distributions u such that

$$\lim_{\lambda \rightarrow \infty} \|\theta(\lambda D)u\|_{L^\infty} = 0 \quad \text{for any } \theta \in \mathcal{D}(\mathbb{R}^3),$$

where $\mathcal{D}(\mathbb{R}^3)$ is the space of smooth compactly supported functions on \mathbb{R}^3 . Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_k \Delta_j u \equiv 0 \quad \text{if } |k - j| \geq 2 \quad \text{and} \quad \Delta_k (S_{j-1} u \Delta_j u) \equiv 0 \quad \text{if } |k - j| \geq 5.$$

We recall now the definitions of homogeneous Besov spaces and Chemin-Lerner-type spaces $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s(\mathbb{R}^3))$ from [8].

Definition 2.1. Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'_h(\mathbb{R}^3)$, we set

$$\|u\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \{2^{qs} \|\Delta_q u\|_{L^p}\}_{\ell^r}.$$

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $\dot{B}_{p,r}^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'_h(\mathbb{R}^3) \mid \|u\|_{\dot{B}_{p,r}^s} < \infty\}$.
- If $k \in \mathbb{N}$ and $\frac{3}{p} + k \leq s < \frac{3}{p} + k + 1$ (or $s = \frac{3}{p} + k + 1$ if $r = 1$), then $\dot{B}_{p,r}^s(\mathbb{R}^3)$ is defines as the subset of distributions $u \in \mathcal{S}'_h(\mathbb{R}^3)$ such that $\partial^\beta u \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^3)$ whenever $|\beta| = k$.

Definition 2.2. Let $s \in \mathbb{R}$, $(r, \lambda, p) \in [1, +\infty]^3$ and $T \in (0, +\infty]$. We define $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s(\mathbb{R}^3))$ as the completion of $C([0, T]; \mathcal{S}(\mathbb{R}^3))$ by the norm

$$\|f\|_{\tilde{L}_T^\lambda(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left(\sum_{q \in \mathbb{Z}} 2^{qrs} \left(\int_0^T \|\Delta_q f(t)\|_{L^p}^\lambda dt \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s)$.

As we shall frequently use the anisotropic Bernstein inequalities. For the convenience of the reader, we recall the following Bernstein type lemma from [8, 25].

Lemma 2.1. *Let \mathcal{C} be a ring of \mathbb{R}^3 and $N \in \mathbb{N}$. There exists a constant C such that for any homogeneous function σ of degree m smooth outside of 0 and all $1 \leq a \leq b \leq \infty$, we have
If the support of \hat{u} is included in $2^k \mathcal{C}$, then*

$$C^{-1-N} 2^{kN} \|u\|_{L^a} \leq \sup_{|\alpha|=N} \|\partial^\alpha u\|_{L^a} \leq C^{1+N} 2^{kN} \|u\|_{L^a}.$$

If the support of \hat{u} is included in $2^k \mathcal{C}$, then

$$\|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} 2^{km+3k(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}.$$

*Furthermore, let \mathcal{B}_h (resp. \mathcal{B}_v) a ball of \mathbb{R}_h^2 (resp. \mathbb{R}_v), let $1 \leq p_2 \leq p_1 \leq \infty$, $1 \leq q_2 \leq q_1 \leq \infty$, we have
If the support of \hat{u} is included in $2^j \mathcal{B}_h$, then*

$$\|\partial_{x_h}^\alpha u\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{j(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|u\|_{L_h^{p_2}(L_v^{q_1})}.$$

If the support of \hat{u} is included in $2^\ell \mathcal{B}_v$, then

$$\|\partial_{x_3}^\beta u\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))} \|u\|_{L_h^{p_1}(L_v^{q_2})}.$$

In the sequel, we shall frequently use Bony's decomposition from [8] in the homogeneous context:

$$uv = T_u v + T_v u + R(u, v) \quad \text{or} \quad uv = T_u v + \mathcal{R}(u, v), \quad (2.1)$$

where

$$\begin{aligned} T_u v &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad \mathcal{R}(u, v) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j u S_{j+2} v, \\ R(u, v) &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v, \quad \text{and} \quad \tilde{\Delta}_j v \stackrel{\text{def}}{=} \sum_{|j'-j| \leq 1} \Delta_{j'} v. \end{aligned}$$

Finally, for the sake of completeness, we recall the following product laws from Lemma 2.2 in [27]:

Lemma 2.2. *Let $p_2 \geq p_1 \geq 1$, $s_1 \leq \frac{3}{p_1}$, $s_2 \leq \frac{3}{p_2}$ with $s_1 + s_2 > 3 \max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1)$, $a \in \dot{B}_{p_1,1}^{s_1}(\mathbb{R}^3)$, $b \in \dot{B}_{p_2,1}^{s_2}(\mathbb{R}^3)$. Then $ab \in \dot{B}_{p_2,1}^{s_1+s_2-\frac{3}{p_1}}(\mathbb{R}^3)$, and there holds*

$$\|ab\|_{\dot{B}_{p_2,1}^{s_1+s_2-\frac{3}{p_1}}} \lesssim \|a\|_{\dot{B}_{p_1,1}^{s_1}} \|b\|_{\dot{B}_{p_2,1}^{s_2}}.$$

In this paper, however, we need more general product laws. As an application of Littlewood-Paley theory, we only present the following product laws in Besov spaces, which will be used in the sequel. One may check [8] for more general product laws in this respect.

Lemma 2.3. *Let $1 \leq p \leq q$, $s \leq \frac{3}{q}$ with $s + \frac{3}{q} > 3 \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}$, $a \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$, $b \in \dot{B}_{p,1}^s(\mathbb{R}^3)$. Then $ab \in \dot{B}_{p,1}^s(\mathbb{R}^3)$, and there holds*

$$\|ab\|_{\dot{B}_{p,1}^s} \lesssim \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{p,1}^s}.$$

Proof. By applying Bony's decomposition, we can get that

$$ab = T_a b + T_b a + R(a, b).$$

We can obtain by using Lemma 2.1 that

$$\begin{aligned} \|\Delta_j(T_a b)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 5} \|S_{j'-1} a\|_{L^\infty} \|\Delta_{j'} b\|_{L^p} \\ &\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} 2^{j'' \frac{3}{q}} \|\Delta_{j''} a\|_{L^q} \|\Delta_{j'} b\|_{L^p} \\ &\lesssim d_j 2^{-js} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{p,1}^s}, \end{aligned}$$

and for $1 \leq p \leq q$, $s \leq \frac{3}{q}$, we get by using Lemma 2.1 once again that

$$\|\Delta_j(T_b a)\|_{L^p} \lesssim \sum_{|j'-j| \leq 5} \|S_{j'-1} b\|_{L^{\frac{pq}{q-p}}} \|\Delta_{j'} a\|_{L^q} \lesssim d_j 2^{-js} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{p,1}^s}.$$

Then, in the case where $\frac{1}{p} + \frac{1}{q} > 1$, and notice that $s > \frac{3}{p} - 3$, we obtain

$$\begin{aligned} \|\Delta_j R(a, b)\|_{L^p} &\lesssim 2^{j(3-\frac{3}{p})} \sum_{j' \geq j-N_0} \|\Delta_{j'} a \tilde{\Delta}_{j'} b\|_{L^1} \\ &\lesssim 2^{j(3-\frac{3}{p})} \sum_{j' \geq j-N_0} \|\Delta_{j'} a\|_{L^{p'}} \|\tilde{\Delta}_{j'} b\|_{L^p} \\ &\lesssim 2^{j(3-\frac{3}{p})} \sum_{j' \geq j-N_0} 2^{j'(\frac{3}{q}-3+\frac{3}{p})} \|\Delta_{j'} a\|_{L^q} \|\tilde{\Delta}_{j'} b\|_{L^p} \\ &\lesssim d_j 2^{-js} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{p,1}^s}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. If $\frac{1}{p} + \frac{1}{q} \leq 1$, we use the following estimate

$$\|\Delta_j R(a, b)\|_{L^p} \lesssim 2^{j\frac{3}{q}} \sum_{j' \geq j - N_0} \|\Delta_{j'} a\|_{L^q} \|\tilde{\Delta}_{j'} b\|_{L^p} \lesssim d_j 2^{-js} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|b\|_{\dot{B}_{p,1}^s}$$

for $s + \frac{3}{q} > 0$. So we can deduce the proof of the lemma from the above estimates. \square

And we also shall frequently use the following version of Gagliardo-Nirenberg inequality [24]: for any $1 \leq p \leq \infty$,

$$\|a\|_{L^\infty(\mathbb{R})} \leq C \|a\|_{L^p(\mathbb{R})}^{1-\frac{1}{p}} \|\nabla a\|_{L^p(\mathbb{R})}^{\frac{1}{p}}, \quad \text{when } a \in \mathcal{D}(\mathbb{R}). \quad (2.2)$$

As applications of (2.2), the following estimates shall be used frequently in this article.

Lemma 2.4. *Let $1 < p \leq m, r \leq \infty$, $u = (u^h, u^3) \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$ with $\text{div} u = 0$. Then there hold that*

$$\|\Delta_j u^3\|_{L_t^2(L_h^m(L_v^r))} \lesssim d_j 2^{-j(\frac{2}{m}+\frac{1}{r})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}+\frac{1}{r}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{\frac{1}{p}-\frac{1}{r}}, \quad (2.3)$$

$$\|\Delta_j u^3\|_{L_t^1(L_h^m(L_v^r))} \lesssim d_j 2^{-j(1+\frac{2}{m}+\frac{1}{r})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}+\frac{1}{r}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}-\frac{1}{r}}. \quad (2.4)$$

Proof. By applying Lemma 2.1, Gagliardo-Nirenberg inequality and $\partial_3 u^3 = -\text{div}_h u^h$, we have

$$\begin{aligned} \|\Delta_j u^3\|_{L_t^2(L_h^m(L_v^r))} &\lesssim 2^{-j(\frac{2}{m}-\frac{2}{p})} \|\Delta_j u^3\|_{L_t^2(L_h^p(L_v^r))} \\ &\lesssim 2^{-j(\frac{2}{m}-\frac{2}{p})} \|\Delta_j u^3\|_{L_t^2(L^p)}^{1-\frac{1}{p}+\frac{1}{r}} \|\partial_3(\Delta_j u^3)\|_{L_t^2(L^p)}^{\frac{1}{p}-\frac{1}{r}} \\ &\lesssim d_j 2^{-j(\frac{2}{m}+\frac{1}{r})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}+\frac{1}{r}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{\frac{1}{p}-\frac{1}{r}}. \end{aligned}$$

Similarly, we have (2.4). \square

3 The proof of Theorem 1.3 for $p \leq q$

At first, we give the estimates of the transport equation. We consider the following free transport equation:

$$\partial_t a + u \cdot \nabla a = 0, \quad a|_{t=0} = a_0. \quad (3.1)$$

Proposition 3.1. *Let $1 < p \leq q$, $u = (u^h, u^3) \in \tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L_T^1(\dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$ with $\text{div} u = 0$ and $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$. Then (3.1) has a unique solution $a \in \mathcal{C}([0, T]; \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$ so that*

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + C \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \left\{ \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}} \right\} \quad (3.2)$$

for any $t \in [0, T]$.

Proof. The existence and uniqueness of solutions to (3.1) essentially follow from the estimate (3.2) for some appropriate solutions to (3.1). For simplicity, here we just present the estimate (3.2) for smooth enough solutions of (3.1). In this case, applying Bony's decomposition (2.1), we obtain

$$u \cdot \nabla a = T_u \nabla a + \mathcal{R}(u, \nabla a).$$

Applying Δ_j to the above equation and taking L^2 inner product of the resulting equation with $|\Delta_j a|^{q-2} \Delta_j a$ (when $q \in (1, 2)$, we need to make some modification as Proposition 2.1 in [11]), we obtain

$$\frac{1}{q} \frac{d}{dt} \|\Delta_j a(t)\|_{L^q}^q + (\Delta_j(T_u \nabla a) | |\Delta_j a|^{q-2} \Delta_j a) + (\Delta_j \mathcal{R}(u, \nabla a) | |\Delta_j a|^{q-2} \Delta_j a) = 0. \quad (3.3)$$

And one can get by using a standard commutator's argument that

$$\begin{aligned} (\Delta_j(T_u \nabla a) \mid |\Delta_j a|^{q-2} \Delta_j a) &= \sum_{|j'-j| \leq 5} \{([\Delta_j; S_{j'-1} u] \Delta_{j'} \nabla a \mid |\Delta_j a|^{q-2} \Delta_j a) \\ &\quad + ((S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_{j'} \nabla a \mid |\Delta_j a|^{q-2} \Delta_j a)\}. \end{aligned}$$

Then thanks to (3.3) and using an argument for the L^q energy estimate in [11], we arrive at

$$\begin{aligned} \|\Delta_j a(t)\|_{L^q} &\leq \|\Delta_j a_0\|_{L^q} + C \int_0^t \left\{ \sum_{|j'-j| \leq 5} (\|[\Delta_j; S_{j'-1} u] \Delta_{j'} \nabla a(t')\|_{L^q} \right. \\ &\quad \left. + \|(S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_{j'} \nabla a(t')\|_{L^q}) + \|\Delta_j \mathcal{R}(u, \nabla a)(t')\|_{L^q} \right\} dt'. \end{aligned} \quad (3.4)$$

We first get by applying the classical estimate on commutators (see [8].) and (2.4) with $m = r = \infty$ that

$$\begin{aligned} &\sum_{|j'-j| \leq 5} \|[\Delta_j; S_{j'-1} u] \Delta_{j'} \nabla a\|_{L_t^1(L^q)} \\ &\lesssim \sum_{|j'-j| \leq 5} (\|S_{j'-1} \nabla u^h\|_{L_t^1(L^\infty)} \|\Delta_{j'} a\|_{L_t^\infty(L^q)} + \|S_{j'-1} \nabla u^3\|_{L_t^1(L^\infty)} \|\Delta_{j'} a\|_{L_t^\infty(L^q)}) \\ &\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} (\|\Delta_{j''} \nabla u^h\|_{L_t^1(L^\infty)} \|\Delta_{j'} a\|_{L_t^\infty(L^q)} + 2^{j''} \|\Delta_{j''} u^3\|_{L_t^1(L^\infty)} \|\Delta_{j'} a\|_{L_t^\infty(L^q)}) \\ &\lesssim d_j 2^{-j \frac{3}{q}} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} (\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{\frac{1}{p}}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} &\sum_{|j'-j| \leq 5} \|(S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_{j'} \nabla a\|_{L_t^1(L^q)} \\ &\lesssim \sum_{|j'-j| \leq 5} (\|S_{j'-1} \nabla u^h - S_{j-1} \nabla u^h\|_{L_t^1(L^\infty)} \|\Delta_j a\|_{L_t^\infty(L^q)} \\ &\quad + \|S_{j'-1} \nabla u^3 - S_{j-1} \nabla u^3\|_{L_t^1(L^\infty)} \|\Delta_j a\|_{L_t^\infty(L^q)}) \\ &\lesssim d_j 2^{-j \frac{3}{q}} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} (\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{\frac{1}{p}}). \end{aligned}$$

For $1 < p \leq q$, thanks to Lemma 2.1 and (2.4) with $m = q, r = \infty$, we obtain

$$\begin{aligned} &\|\Delta_j \mathcal{R}(u, \nabla a)\|_{L_t^1(L^q)} \\ &\lesssim \sum_{j' \geq j-N_0} (\|S_{j'+2} \nabla_h a\|_{L_t^\infty(L^\infty)} \|\Delta_{j'} u^h\|_{L_t^1(L^q)} + \|S_{j'+2} \partial_3 a\|_{L_t^\infty(L_h^\infty(L_v^q))} \|\Delta_{j'} u^3\|_{L_t^1(L_h^q(L_v^\infty))}) \\ &\lesssim \sum_{j' \geq j-N_0} \sum_{j'' \leq j'+1} (2^{j''(1+\frac{3}{q})} \|\Delta_{j''} a\|_{L_t^\infty(L^q)} 2^{j'(\frac{3}{p}-\frac{3}{q})} \|\Delta_{j'} u^h\|_{L_t^1(L^p)} \\ &\quad + 2^{j''(1+\frac{2}{q})} \|\Delta_{j''} a\|_{L_t^\infty(L^q)} \|\Delta_{j'} u^3\|_{L_t^1(L_h^q(L_v^\infty))}) \\ &\lesssim d_j 2^{-j \frac{3}{q}} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} (\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{\frac{1}{p}}). \end{aligned}$$

Substituting the above estimates into (3.4) and taking summation for $j \in \mathbb{Z}$, we conclude the proof of (3.2). \square

As we all known, deriving the estimate for the pressure term is the main difficulty in the study of the well-posedness of incompressible inhomogeneous Navier-Stokes equations. In the following, our goal is to provide the estimates for the pressure term. We first get taking div to the momentum equation of (1.2) that

$$-\Delta \Pi = \operatorname{div}(a \nabla \Pi) - \mu \operatorname{div}(a \Delta u) + \sum_{i,j=1}^2 \partial_i \partial_j (u^i u^j) + 2 \partial_3 \operatorname{div}_h (u^3 u^h) - 2 \partial_3 (u^3 \operatorname{div}_h u^h), \quad (3.5)$$

where, for a vector field $u = (u^h, u^3)$, we denote $\operatorname{div}_h u^h = \partial_1 u^1 + \partial_2 u^2$.

The following proposition concerning the estimate of the pressure will be the main ingredient used in the estimate of u^h and u^3 . Denote

$$\begin{aligned} A(a, u) &\stackrel{\text{def}}{=} \mu \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} (\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}) \\ &\quad + \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}} \\ &\quad + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\alpha} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\alpha}. \end{aligned} \quad (3.6)$$

Proposition 3.2. *Let $1 < p \leq q < 6$ with $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{3}$, $a \in \tilde{L}_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$ and $u \in \tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L_T^1(\dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$. Then (3.5) has a unique solution $\nabla \Pi \in L_T^1(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3))$ which decays to zero when $|x| \rightarrow \infty$ so that for all $t \in [0, T]$, there holds*

$$\|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \leq CA(a, u), \quad (3.7)$$

provided that $C\|a\|_{\tilde{L}_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \frac{1}{2}$, and α is defined as (1.6).

The proof of this proposition will mainly be based on the following lemmas :

Lemma 3.1. *Let $p > 1$, under the assumptions of Proposition 3.2, one has*

$$\begin{aligned} \|\Delta_j(u^3 u^h)\|_{L_t^1(L^p)} &\lesssim d_j 2^{-j\frac{3}{p}} (\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}} \\ &\quad + \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}}). \end{aligned}$$

for all $t \leq T$.

Proof. We first get by applying Bony's decomposition (2.1) that

$$u^3 u^h = T_{u^3} u^h + T_{u^h} u^3 + R(u^3, u^h). \quad (3.8)$$

Applying Lemma 2.1 and (2.3) with $m = r = \infty$ gives rise to

$$\begin{aligned} \|\Delta_j(T_{u^3} u^h)\|_{L_t^1(L^p)} &\lesssim \sum_{|j'-j| \leq 5} \|S_{j'-1} u^3\|_{L_t^2(L^\infty)} \|\Delta_{j'} u^h\|_{L_t^2(L^p)} \\ &\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} \|\Delta_{j''} u^3\|_{L_t^2(L^\infty)} \|\Delta_{j'} u^h\|_{L_t^2(L^p)} \\ &\lesssim d_j 2^{-j\frac{3}{p}} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}}. \end{aligned}$$

And while (2.4) applied with $m = p, r = \infty$ gives

$$\begin{aligned} \|\Delta_j(T_{u^h} u^3)\|_{L_t^1(L^p)} &\lesssim \sum_{|j'-j| \leq 5} \|S_{j'-1} u^h\|_{L_t^\infty(L_h^\infty(L_v^p))} \|\Delta_{j'} u^3\|_{L_t^1(L_h^p(L_v^\infty))} \\ &\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} 2^{j''\frac{2}{p}} \|\Delta_{j''} u^h\|_{L_t^\infty(L^p)} \|\Delta_{j'} u^3\|_{L_t^1(L_h^p(L_v^\infty))} \\ &\lesssim d_j 2^{-j\frac{3}{p}} \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}}, \end{aligned}$$

and

$$\begin{aligned}
\|\Delta_j R(u^3, u^h)\|_{L_t^1(L^p)} &\lesssim \sum_{j' \geq j-N_0} \|\Delta_{j'} u^3\|_{L_t^1(L_h^p(L_v^\infty))} \|\tilde{\Delta}_{j'} u^h\|_{L_t^\infty(L_h^\infty(L_v^p))} \\
&\lesssim \sum_{j' \geq j-N_0} 2^{j' \frac{2}{p}} \|\Delta_{j'} u^3\|_{L_t^1(L_h^p(L_v^\infty))} \|\tilde{\Delta}_{j'} u^h\|_{L_t^\infty(L^p)} \\
&\lesssim d_j 2^{-j \frac{3}{p}} \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}}.
\end{aligned}$$

Along with (3.8), we prove the inequality of Lemma 3.1. \square

Lemma 3.2. *Under the assumptions of Proposition 3.2, when $1 < p < 6$, one has*

$$\|\Delta_j(u^3 \operatorname{div}_h u^h)\|_{L_t^1(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})} (\|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}} + \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\alpha} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\alpha}),$$

whereas

$$\alpha = \begin{cases} \frac{1}{p}, & 1 < p < 5 \\ \varepsilon, & 5 \leq p < 6 \end{cases}$$

for $0 < \varepsilon < \frac{6}{p} - 1$.

Proof. We first get by applying Bony's decomposition (2.1) that

$$u^3 \operatorname{div}_h u^h = T_{u^3} \operatorname{div}_h u^h + T_{\operatorname{div}_h u^h} u^3 + R(u^3, \operatorname{div}_h u^h). \quad (3.9)$$

Applying (2.3) with $m = r = \infty$, we obtain

$$\begin{aligned}
\|\Delta_j(T_{u^3} \operatorname{div}_h u^h)\|_{L_t^1(L^p)} &\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} \|\Delta_{j''} u^3\|_{L_t^2(L^\infty)} \|\Delta_{j'}(\operatorname{div}_h u^h)\|_{L_t^2(L^p)} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}},
\end{aligned}$$

and similarly, we get

$$\begin{aligned}
\|\Delta_j(T_{\operatorname{div}_h u^h} u^3)\|_{L_t^1(L^p)} &\lesssim \sum_{|j'-j| \leq 5} \|S_{j'-1}(\operatorname{div}_h u^h)\|_{L_t^2(L_h^\infty(L_v^p))} \|\Delta_{j'} u^3\|_{L_t^2(L_h^p(L_v^\infty))} \\
&\lesssim \sum_{|j'-j| \leq 5} \sum_{j'' \leq j'-2} 2^{j'' \frac{2}{p}} \|\Delta_{j''}(\operatorname{div}_h u^h)\|_{L_t^2(L^p)} \|\Delta_{j'} u^3\|_{L_t^2(L_h^p(L_v^\infty))} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}}.
\end{aligned}$$

For $1 < p < 5$, while (2.3) applied with $m = p, r = \infty$ gives

$$\begin{aligned}
\|\Delta_j(R(u^3, \operatorname{div}_h u^h))\|_{L_t^1(L^p)} &\lesssim 2^{j \frac{2}{p}} \sum_{j' \geq j-N_0} \|\tilde{\Delta}_{j'}(\operatorname{div}_h u^h) \Delta_{j'} u^3\|_{L_t^1(L_h^{\frac{p}{2}}(L_v^p))} \\
&\lesssim 2^{j \frac{2}{p}} \sum_{j' \geq j-N_0} \|\tilde{\Delta}_{j'}(\operatorname{div}_h u^h)\|_{L_t^2(L^p)} \|\Delta_{j'} u^3\|_{L_t^2(L_h^p(L_v^\infty))} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}}.
\end{aligned}$$

For $5 \leq p < 6$, we get by using (2.3) with $m = p, \frac{1}{r} = \frac{1}{p} - \varepsilon$ that

$$\begin{aligned}
\|\Delta_j(R(u^3, \operatorname{div}_h u^h))\|_{L_t^1(L^p)} &\lesssim 2^{j(\frac{3}{p}-\varepsilon)} \sum_{j' \geq j-N_0} \|\tilde{\Delta}_{j'}(\operatorname{div}_h u^h) \Delta_{j'} u^3\|_{L_t^1(L_h^{\frac{p}{2}}(L_v^{1/(\frac{2}{p}-\varepsilon)}))} \\
&\lesssim 2^{j(\frac{3}{p}-\varepsilon)} \sum_{j' \geq j-N_0} \|\tilde{\Delta}_{j'}(\operatorname{div}_h u^h)\|_{L_t^2(L^p)} \|\Delta_{j'} u^3\|_{L_t^2(L_h^p(L_v^{1/(\frac{1}{p}-\varepsilon)}))} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\varepsilon} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\varepsilon}
\end{aligned}$$

where $1 - \frac{6}{p} + \varepsilon < 0$. Thus again thanks to (3.9), we conclude the proof of Lemma 3.2. \square

Proof of proposition 3.2: Again as both the proof of the existence and uniqueness of solutions to (3.5) is essentially followed by the estimates (3.7) for some appropriate approximate solutions of (3.5). For simplicity, we just prove (3.7) for smooth enough solutions of (3.5). Indeed thanks to (3.5) and $\operatorname{div} u = 0$, we have

$$\begin{aligned} \nabla \Pi &= \nabla(-\Delta)^{-1}[\operatorname{div}(a\nabla \Pi) + \sum_{i,j=1}^2 \partial_i \partial_j (u^i u^j) + 2\partial_3 \sum_{i=1}^2 \partial_i (u^3 u^i) \\ &\quad - 2\partial_3 (u^3 \operatorname{div}_h u^h) - \mu \operatorname{div}_h (a\Delta u^h) - \mu \partial_3 (a\Delta u^3)]. \end{aligned} \quad (3.10)$$

Applying Δ_j to the above equation and using Lemma 2.1 leads to

$$\begin{aligned} \|\Delta_j(\nabla \Pi)\|_{L_t^1(L^p)} &\lesssim \|\Delta_j(a\nabla \Pi)\|_{L_t^1(L^p)} + 2^j(\|\Delta_j(u^h \otimes u^h)\|_{L_t^1(L^p)} + \|\Delta_j(u^3 u^h)\|_{L_t^1(L^p)}) \\ &\quad + \|\Delta_j(u^3 \operatorname{div}_h u^h)\|_{L_t^1(L^p)} + \mu \|\Delta_j(a\Delta u^h)\|_{L_t^1(L^p)} + \mu \|\Delta_j(a\Delta u^3)\|_{L_t^1(L^p)}. \end{aligned} \quad (3.11)$$

For $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{3}$, applying Lemma 2.3 gives rise to

$$\begin{aligned} \|\Delta_j(a\nabla \Pi)\|_{L_t^1(L^p)} &\lesssim d_j 2^{j(1-\frac{3}{p})} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \quad \text{and} \\ \|\Delta_j(u^h \otimes u^h)\|_{L_t^1(L^p)} &\lesssim d_j 2^{-j\frac{3}{p}} \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}, \\ \|\Delta_j(a\Delta u^h)\|_{L_t^1(L^p)} &\lesssim d_j 2^{j(1-\frac{3}{p})} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}, \\ \|\Delta_j(a\Delta u^3)\|_{L_t^1(L^p)} &\lesssim d_j 2^{j(1-\frac{3}{p})} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}, \end{aligned}$$

which along with Lemma 3.1, Lemma 3.2 and (3.11) implies that

$$\|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \leq C\{ \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + A(a, u) \}$$

for all $t \leq T$. So provided that $C\|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \frac{1}{2}$, we conclude the proof of (3.7). \square

Motivated by [18, 26, 27, 29], and based on the estimate of the pressure, we shall deal with the L^p type energy estimate for u^h and u^3 separately.

Proposition 3.3. *Under the assumption of Proposition 3.2 and*

$$C\|a\|_{\tilde{L}_T^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \frac{1}{2} \quad (3.12)$$

there holds

$$\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + CA(a, u). \quad (3.13)$$

Proof. According to the second equation of (1.2), we have

$$\partial_t u^h + u \cdot \nabla u^h + (1+a)(\nabla_h \Pi - \mu \Delta u^h) = 0.$$

Applying the operator Δ_j to the above equation and taking the L^2 inner product of the resulting equation with $|\Delta_j u^h|^{p-2} \Delta_j u^h$ (when $p \in (1, 2)$, we need to make some modification as Proposition 2.1 in [11]), we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\Delta_j u^h\|_{L^p}^p - \mu \int_{\mathbb{R}^3} \Delta \Delta_j u^h |\Delta_j u^h|^{p-2} \Delta_j u^h dx &= - \int_{\mathbb{R}^3} \{ \Delta_j (u \cdot \nabla u^h) \\ &\quad + \Delta_j ((1+a) \nabla_h \Pi) - \mu \Delta_j (a \Delta u^h) \} |\Delta_j u^h|^{p-2} \Delta_j u^h dx. \end{aligned} \quad (3.14)$$

However thanks to Lemma A.5 of Appendix in [11], there exists a positive constant \bar{c} so that

$$-\int_{\mathbb{R}^3} \Delta \Delta_j u^h |\Delta_j u^h|^{p-2} \Delta_j u^h dx \geq \bar{c} 2^{2j} \|\Delta_j u^h\|_{L^p}^p,$$

so we get from (3.14) that

$$\begin{aligned} & \frac{d}{dt} \|\Delta_j u^h\|_{L^p} + \bar{c} \mu 2^{2j} \|\Delta_j u^h\|_{L^p} \\ & \leq \|\Delta_j(u \cdot \nabla u^h)\|_{L^p} + \|\Delta_j((1+a)\nabla_h \Pi)\|_{L^p} + \mu \|\Delta_j(a\Delta u^h)\|_{L^p}, \end{aligned} \quad (3.15)$$

and integrating in time, we get

$$\begin{aligned} \|\Delta_j u^h\|_{L_t^\infty(L^p)} + \bar{c} \mu 2^{2j} \|\Delta_j u^h\|_{L_t^1(L^p)} & \leq \|\Delta_j u_0^h\|_{L^p} + \|\Delta_j(u \cdot \nabla u^h)\|_{L_t^1(L^p)} \\ & \quad + \|\Delta_j((1+a)\nabla_h \Pi)\|_{L_t^1(L^p)} + \mu \|\Delta_j(a\Delta u^h)\|_{L_t^1(L^p)}. \end{aligned} \quad (3.16)$$

Applying Lemma 2.3 and Lemma 3.1, we obtain

$$\begin{aligned} \|\Delta_j(u \cdot \nabla u^h)\|_{L_t^1(L^p)} & \leq \|\Delta_j \operatorname{div}_h(u^h \otimes u^h)\|_{L_t^1(L^p)} + \|\Delta_j(\partial_3(u^3 u^h))\|_{L_t^1(L^p)} \\ & \lesssim 2^j (\|\Delta_j(u^h \otimes u^h)\|_{L_t^1(L^p)} + \|\Delta_j(u^3 u^h)\|_{L_t^1(L^p)}) \\ & \lesssim d_j 2^{j(1-\frac{3}{p})} (\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1-\frac{1}{p}} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}^{1+\frac{1}{p}} \\ & \quad + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{1-\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}}})^{\frac{1}{p}}). \end{aligned}$$

While applying Lemma 2.3 and Proposition 3.2, under the assumption (3.12), we arrive at

$$\begin{aligned} \|\Delta_j((1+a)\nabla_h \Pi)\|_{L_t^1(L^p)} & \lesssim d_j 2^{j(1-\frac{3}{p})} (1 + \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})}) \|\nabla_h \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \\ & \lesssim d_j 2^{j(1-\frac{3}{p})} A(a, u) \end{aligned}$$

and

$$\|\Delta_j(a\Delta u^h)\|_{L_t^1(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}.$$

Substituting the above estimates into (3.16) and the condition (3.12), we deduce the proof of (3.13). \square

Proposition 3.4. *Under the assumption of Proposition 3.2, we have*

$$\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c} \mu \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + C\{\|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})} + A(a, u)\}. \quad (3.17)$$

Proof. According to the second equation of (1.2), we have

$$\partial_t u^3 + u \cdot \nabla u^3 + (1+a)(\partial_3 \Pi - \mu \Delta u^3) = 0,$$

we get by a similar derivation of (3.15) that

$$\begin{aligned} \|\Delta_j u^3(t)\|_{L^p} + \bar{c} \mu 2^{2j} \|\Delta_j u^3\|_{L_t^1(L^p)} & \leq \|\Delta_j u_0^3\|_{L^p} + \|\Delta_j(u \cdot \nabla u^3)\|_{L_t^1(L^p)} \\ & \quad + \|\Delta_j((1+a)\partial_3 \Pi)\|_{L_t^1(L^p)} + \mu \|\Delta_j(a\Delta u^3)\|_{L_t^1(L^p)}. \end{aligned} \quad (3.18)$$

By using Lemma 2.1 and Lemma 2.3, we deduce that

$$\begin{aligned} \|\Delta_j(u \cdot \nabla u^3)\|_{L_t^1(L^p)} & \lesssim 2^j \|\Delta_j(u^h u^3)\|_{L_t^1(L^p)} + \|\Delta_j(u^3 \operatorname{div}_h u^h)\|_{L_t^1(L^p)} \\ & \lesssim d_j 2^{j(1-\frac{3}{p})} \|u^3\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})} \|u^h\|_{\tilde{L}_t^2(\dot{B}_{p,1}^{\frac{3}{p}})}. \end{aligned}$$

We get by using Proposition 3.2 and (3.12) that

$$\|\Delta_j((1+a)\partial_3\Pi)\|_{L_t^1(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})} A(a, u)$$

and

$$\|\Delta_j(a\Delta u^3)\|_{L_t^1(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}.$$

Then we obtain (3.17) by substituting the above estimates into (3.18). \square

The proof of Theorem 1.3 for $p \leq q$: Indeed given $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$ and $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ with $\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}$ sufficiently small and p, q satisfying the conditions listed in Theorem 1.3 for $p \leq q$, Theorem 1.1 ensures that there exists a positive time T so that the system (1.2) has a unique solution $(a, u, \nabla\Pi) \in E_{p,q,T}$. We denote T^* to be the largest existence time. Hence to prove Theorem 1.3 for $p \leq q$, we only need to prove that $T^* = \infty$.

Now let η be a small enough positive constant, which will be determined later on. We define \mathcal{T} by

$$\begin{aligned} \mathcal{T} &\stackrel{def}{=} \sup\{t \in [0, T^*) : \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq 4\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}, \\ &\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq 4\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu, \\ &\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq 4\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu\}. \end{aligned} \quad (3.19)$$

In what follows, we shall prove that $\mathcal{T} = T^*$ under the assumptions of (1.4)-(1.5).

If not, we assume that

$$\mathcal{T} < T^*.$$

And for $t \leq \mathcal{T}$, we deduce from (3.2) that

$$\begin{aligned} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} &\leq \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \frac{C}{\bar{c}} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \left\{ \frac{4}{\mu} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta \right. \\ &\quad \left. + \left(\frac{4}{\mu} \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + 1 \right)^{1-\frac{1}{p}} \left(\frac{4}{\mu} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta \right)^{\frac{1}{p}} \right\}. \end{aligned}$$

By taking

$$\frac{C}{\bar{c}} \left\{ \frac{4}{\mu} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta + \left(\frac{4}{\mu} \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + 1 \right)^{1-\frac{1}{p}} \left(\frac{4}{\mu} \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta \right)^{\frac{1}{p}} \right\} < \frac{1}{2}, \quad (3.20)$$

we obtain

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq 2\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \quad (3.21)$$

for $t \leq \mathcal{T}$. Thanks to (3.17), on the one hand, we obtain

$$\begin{aligned} &\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \\ &\leq \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \frac{C}{\bar{c}\mu} (\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}) \\ &\quad \times (4\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu) + C\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (4\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu) + C\mu\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \\ &\quad + \frac{C}{\bar{c}\mu} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^{1+\frac{1}{p}} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\frac{1}{p}} + \frac{C}{\bar{c}\mu} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^2 \\ &\quad + \frac{C}{\bar{c}\mu} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^{1+\alpha} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\alpha} \end{aligned}$$

for $t \leq \mathcal{T}$, and taking

$$\begin{aligned} & \frac{C}{\bar{c}\mu} \{ (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^{\frac{1}{p}} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\frac{1}{p}} + (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^\alpha \\ & \times (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\alpha} + (4\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu) \} \leq \frac{1}{8} \end{aligned} \quad (3.22)$$

and combining with

$$C\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \leq \frac{1}{8}\bar{c}, \quad (3.23)$$

we obtain

$$\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq 2\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + 2(\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu).$$

While taking

$$\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu \leq \frac{\mu}{4}, \quad (3.24)$$

we get that

$$\|u^3\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq 2\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \frac{\mu}{2}. \quad (3.25)$$

On the other hand, for $t \leq \mathcal{T}$, we can deduce from (3.13) that

$$\begin{aligned} & \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + C(\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \\ & + \frac{C}{\bar{c}\mu}\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^{\frac{1}{p}} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\frac{1}{p}} \\ & + C\mu\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}}\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \frac{C}{\bar{c}}\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu) \\ & + \frac{C}{\bar{c}\mu}\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} (\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \eta\mu)^\alpha (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \mu)^{1-\alpha}. \end{aligned}$$

By taking the conditions (3.22)-(3.23), choosing $\eta = \frac{8C}{\bar{c}}\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} (\|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} / \mu + 1)$, we obtain

$$\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \bar{c}\mu\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} \leq 2\|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \frac{\eta\mu}{4} \quad (3.26)$$

for $t \leq \mathcal{T}$. Combining (3.12), (3.20), (3.22), (3.23) and (3.24), we can reach (3.21), (3.25) and (3.26) if we take C large enough in (1.4)-(1.5). And this contradicts with the definition (3.19), thus we conclude that $\mathcal{T} = T^*$. Then we complete the proof of Theorem 1.3 for $p \leq q$ by standard continuation argument. \square

4 The proof of Theorem 1.3 for $p > q$

For $1 < q < p < 6$ with $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$, $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$, $u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, by using the embedding of Besov spaces, we get that $a_0 \in \dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$. So by the proof of Theorem 1.3 for $p = q$ in Section 3, there exists a unique global solution $(a, u \nabla \Pi) \in E_{p,p}$. Then, by using the same method of Proposition 3.1, we imply $a \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$ in the following.

By applying Bony's decomposition (2.1), we get

$$\begin{aligned} \|\Delta_j a(t)\|_{L^q} & \leq \|\Delta_j a_0\|_{L^q} + C \int_0^t \left\{ \sum_{|j'-j| \leq 5} (\|[\Delta_j; S_{j'-1}u]\Delta_{j'} \nabla a\|_{L^q} \right. \\ & \left. + \|(S_{j'-1}u - S_{j-1}u)\Delta_j \Delta_{j'} \nabla a\|_{L^q}) + \|T_{\nabla a} u\|_{L^q} + \|R(u, \nabla a)\|_{L^q} \right\} dt'. \end{aligned} \quad (4.1)$$

We get by applying Lemma 2.1 and the classical estimate on commutators that

$$\sum_{|j'-j|\leq 5} \|[\Delta_j; S_{j'-1}u]\Delta_{j'}\nabla a\|_{L^q} \lesssim \sum_{|j'-j|\leq 5} \|S_{j'-1}\nabla u\|_{L^\infty} \|\Delta_{j'}a\|_{L^q} \lesssim d_j 2^{-j\frac{3}{q}} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}.$$

Similarly, we obtain

$$\begin{aligned} \sum_{|j'-j|\leq 5} \|(S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'}\nabla a\|_{L^q} &\lesssim \sum_{|j'-j|\leq 5} \|S_{j'-1}\nabla u - S_{j-1}\nabla u\|_{L^\infty} \|\Delta_{j'}a\|_{L^q} \\ &\lesssim d_j 2^{-j\frac{3}{q}} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}. \end{aligned}$$

Because of $q < p$ and $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$, we obtain

$$\begin{aligned} \|T_{\nabla a}u\|_{L^q} &\lesssim \sum_{|j'-j|\leq 5} \|S_{j'-1}\nabla a\|_{L^{\frac{pq}{p-q}}} \|\Delta_{j'}u\|_{L^p} \\ &\lesssim \sum_{|j'-j|\leq 5} \sum_{j''\leq j'-2} 2^{j''(1+\frac{3}{p})} \|\Delta_{j''}a\|_{L^q} \|\Delta_{j'}u\|_{L^p} \\ &\lesssim d_j 2^{-j\frac{3}{q}} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}. \end{aligned}$$

Finally, thanks to $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$, we arrive at

$$\begin{aligned} \|R(u, \nabla a)\|_{L^q} &\lesssim \sum_{j'\geq j-N_0} \|\Delta_{j'}u\|_{L^p} \|\tilde{\Delta}_{j'}\nabla a\|_{L^{\frac{pq}{p-q}}} \\ &\lesssim \sum_{j'\geq j-N_0} 2^{j'(1+\frac{3}{p})} \|\Delta_{j'}u\|_{L^p} \|\tilde{\Delta}_{j'}a\|_{L^q} \\ &\lesssim d_j 2^{-j\frac{3}{q}} \|a\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}}. \end{aligned}$$

Substituting the above estimates into (4.1), we can deduce that

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + C \int_0^t \|a(t')\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \|u(t')\|_{\dot{B}_{p,1}^{1+\frac{3}{p}}} dt'.$$

Applying Gronwall's inequality, we get that

$$\|a(t)\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{3}{q}})} \leq \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} \exp\{C\|u\|_{L^1([0,\infty);\dot{B}_{p,1}^{1+\frac{3}{p}})}\}$$

for any $t > 0$. Therefore we obtain $a \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$. \square

Appendix

In this section, we shall give the proof of Theorem 1.4 briefly. At first, for the convenience of the readers, we recall the definition of multiplier spaces to Besov spaces from [23] and some facts in [8]:

Definition 4.1 (see Chapter 4 in [23]). We call f belonging to the multiplier space, $\mathcal{M}(\dot{B}_{p,1}^s(\mathbb{R}^n))$, of $\dot{B}_{p,1}^s(\mathbb{R}^n)$ if the distributions f satisfies $\Psi f \in \dot{B}_{p,1}^s(\mathbb{R}^n)$ whenever $\Psi \in \dot{B}_{p,1}^s(\mathbb{R}^n)$. We endow this space with the norm

$$\|f\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \stackrel{def}{=} \sup_{\|\Psi\|_{\dot{B}_{p,1}^s}=1} \|\Psi f\|_{\dot{B}_{p,1}^s} \quad \text{for } f \in \mathcal{M}(\dot{B}_{p,1}^s(\mathbb{R}^n)).$$

The estimate of transport equation basically follows from Sect.2 of [19]. Indeed as we shall not use Lagrange approach as that in [14], we need first to investigate the following transport equation:

$$\partial_t a + u \cdot \nabla a = 0, \quad a|_{t=0} = a_0, \quad (4.2)$$

with the initial datum $a_0 \in \mathcal{M}(\dot{B}_{p,1}^s(\mathbb{R}^n))$. We denoted $X_u(t, y)$ to be the flow map determined by u , namely,

$$X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y)) d\tau. \quad (4.3)$$

Lemma 4.1 (Lemma 3.1 in [19]). *Let $s \in (-1, 1)$, $p \geq 1$, $a \in \dot{B}_{p,1}^s(\mathbb{R}^n)$, $u \in L^1((0, T); Lip(\mathbb{R}^n))$, and X_u the flow map determined by (4.3). Then $a \circ X_u \in L^\infty((0, T); \dot{B}_{p,1}^s(\mathbb{R}^n))$, and there holds*

$$\|a \circ X_u\|_{L_t^\infty(\dot{B}_{p,1}^s)} \leq C \|a\|_{\dot{B}_{p,1}^s} \exp\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\}. \quad (4.4)$$

And we obtain the following estimate of $\|a\|_{L_t^\infty(\mathcal{M}(\dot{B}_{p,1}^s))}$ by applying the above Lemma.

Proposition 4.1. *Let $s \in (-1, 1)$ and $p \geq 1$. Let $u \in L_T^1(Lip(\mathbb{R}^3))$ and $a_0 \in \mathcal{M}(\dot{B}_{p,1}^s(\mathbb{R}^3))$. Then (4.2) has a unique solution $a \in L^\infty([0, T]; \mathcal{M}(\dot{B}_{p,1}^s(\mathbb{R}^3)))$ so that*

$$\|a\|_{L_t^\infty(\mathcal{M}(\dot{B}_{p,1}^s))} \leq \|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \exp\{C(\|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})} + \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}})\} \quad (4.5)$$

for any $t \in [0, T]$.

Proof. Thanks to (4.3), we deduce from (4.2) that $a(t, x) = a_0(X_u^{-1}(t, x))$. Then thanks to Definition 4.1 and Lemma 4.1, we obtain

$$\begin{aligned} \|a(t)\|_{\mathcal{M}(\dot{B}_{p,1}^s)} &\leq \sup_{\|\Psi\|_{\dot{B}_{p,1}^s}=1} \|\Psi a(t)\|_{\dot{B}_{p,1}^s} \\ &\leq \sup_{\|\Psi\|_{\dot{B}_{p,1}^s}=1} \|(\Psi \circ X_u(t) a_0) \circ X_u^{-1}(t)\|_{\dot{B}_{p,1}^s} \\ &\leq C \sup_{\|\Psi\|_{\dot{B}_{p,1}^s}=1} \|(\Psi \circ X_u(t)) a_0\|_{\dot{B}_{p,1}^s} \exp\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\} \\ &\leq C \|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \exp\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\} \sup_{\|\Psi\|_{\dot{B}_{p,1}^s}=1} \|\Psi \circ X_u(t)\|_{\dot{B}_{p,1}^s} \\ &\leq C \|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \exp\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\}. \end{aligned}$$

Then we obtain

$$\|a(t)\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \leq C \|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^s)} \exp\{C(\int_0^t \|\nabla u^h(\tau)\|_{L^\infty} d\tau + \int_0^t \|\nabla u^3(\tau)\|_{L^\infty} d\tau)\} \quad (4.6)$$

whereas

$$\int_0^t \|\nabla u^h(\tau)\|_{L^\infty} d\tau \lesssim \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}$$

and

$$\int_0^t \|\nabla u^3(\tau)\|_{L^\infty} d\tau \lesssim \|u^3\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}} \|u^h\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{3}{p}})}^{\frac{1}{p}},$$

using inequality (2.4) with $m = r = \infty, q = \infty$. Thus substituting the above estimates into the inequality (4.6), we complete the proof of Proposition 4.5. \square

Based on the above estimate and the estimate of the pressure obtained by the same method as Proposition 3.2, we complete the proof of Theorem 1.4 by the similar arguments as the proof of Theorem 1.3, and omit the details.

Acknowledgements

This work is partially supported by NSF of China under Grants 11271322, 11331005 and 11271017, National Program for Special Support of Top-Notch Young Professionals, Program for New Century Excellent Talents in University NCET-11-0462, the Fundamental Research Funds for the Central Universities (2012QNA3001).

References

- [1] H. Abidi, Équation de Navier-Stokes avec densité et viscosité variables dans 1' espace critique, *Rev. Mat. Iberoam.* 23 (2007) 537-586.
- [2] H. Abidi, G. L. Gui, P. Zhang. On the decay and stability to global solutions of the 3-D inhomogeneous Navier-Stokes equations, *Comm. Pure Appl. Math.* 64 (2011) 832-881.
- [3] H. Abidi, G. L. Gui, P. Zhang, On the wellposedness of 3-D inhomogeneous Navier-Stokes equations in the critical spaces. *Arch. Ration. Mech. Anal.* 204 (2012) 189-230.
- [4] H. Abidi, G. L. Gui, P. Zhang, Wellposedness of 3-D inhomogeneous Navier-Stokes equations with highly oscillating initial velocity field, *J. Math. Pures Appl.* 100 (2013) 166-203.
- [5] H. Abidi, M. Paicu, Existence globale pour un fluide inhomogène, *Ann. Inst. Fourier (Grenoble)* 57 (2007) 883-917.
- [6] S. N. Antontsev, A. V. Kazhikhov, *Matematicheskie voprosy dinamiki neodnorodnykh zhidkosteĭ.* (Russian) [Mathematical questions of the dynamics of nonhomogeneous fluids] Lecture notes, Novosibirsk State University. Novosibirsk. Gosudarstv. Univ., Novosibirsk, 1973. 121 pp.
- [7] S. N. Antontsev, A. V. Kazhikhov, V. N. Monakov, Boundary value problems in mechanics of nonhomogeneous fluids. *Studies in Mathematics and Its Applications*, 22. North-Holland, Amsterdam, 1990.
- [8] H. Bahouri, J. Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren Math. Wiss. , vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
- [9] J. Bourgain, N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, *J. Func. Anal.* 255 (2008) 2233-2247.
- [10] Q. Chen, C. Miao, Z. Zhang, on the ill-posedness of the compressible Navier-Stokes equations in the critical Besov spaces. *arXiv*. 1109. 6092v1.
- [11] R. Danchin, Local theory in critical spaces for compressible viscous and heat-conducting gases, *Comm. Partial Differential Equations* 26 (2001) 1183-1233.
- [12] R. Danchin, Density-dependent incompressible viscous fluids in critical spaces, *Proc. Roy. Soc. Edinburgh Sect. A* 133 (2003) 1311-1334.
- [13] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids, *Adv. Differential Equations* 9 (2004) 353-386.
- [14] R. Danchin, P. B. Mucha, A Lagrangian approach for the incompressible Navier-Stokes equations with variable density, *Comm. Pure. Appl. Math.* 65 (2012) 1458-1480.
- [15] R. J. Diperna, P. L. Lions, Equations différentielles ordinaires et équations de transport avec des coefficients irréguliers, in: *Séminaire EDP 1988-1989*, Ecole Polytechnique, Palaiseau, 1989.
- [16] H. Fujita, T. Kato, On the Navier-Stokes initial value problem I, *Arch. Ration. Mech. Anal.* 16 (1964) 269-315.
- [17] P. Germain, The second iterate for the Navier-Stokes equation, *J. Func. Anal.* 255 (2008) 2248-2264.

- [18] G. Gui, P. Zhang, Stability to the global solutions of 3-D Navier-Stokes equations, *Adv. Math.* 225 (2010) 1248-1284.
- [19] J. Huang, M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system with rough density, *Progress in Nonlinear Differential Equations and their Applications*, Volume 84 (2013) 159-180.
- [20] A. V. Kazhikov, Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid, *Dokl. Akad. Nauk SSSR* 216 (1974) 1008-1010 (in Russian).
- [21] O. A. Ladyzhenskaya, V. A. Solonnikov, The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids, in: *Boundary Value Problems of Mathematical Physics, and Related Questions of the Theory of Functions*, 8. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 52 (1975) 52-109, 218-219 (in Russian).
- [22] P. L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models*, Oxford Lecture Ser. Math. Appl. , vol.3, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.
- [23] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of Sobolev multipliers. Grundlehren der Mathematischen Wissenschaften*, 337. Springer, Berlin, 2009.
- [24] L. Nirenberg, An extended interpolation inequality, *Ann. Scuola Norm. Sup. Pisa* 20 (1966) 733-737.
- [25] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Matemática Iberoamericana*, 21 (2005) 179-235.
- [26] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible anisotropic Navier-Stokes system in the critical spaces, *Comm. Math. Phys.* 307 (2011) 713-759.
- [27] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system. *J. Funct. Anal.* 262 (2012) 3556-3584.
- [28] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. *SIAM J. Math. Anal.* 21 (1990) 1093-1117.
- [29] T. Zhang, Global wellposedness problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space, *Comm. Math. Phys.* 287 (2009) 211-224.
- [30] T. Zhang, Global strong solutions for equations related to the incompressible viscoelastic fluids with a class of large initial data, *Nonlinear Analysis*, 100 (2014) 59-77.